Dynamical Systems
Tutorial 3: Bifurcations of 1D Maps

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1 Introduction

Up until now we have investigated properties of *flows*, arising from *continuous* dynamical systems - $\frac{dx}{dt} = f(x;\mu)$, $f \in C^r$. Today we will consider a new class of dynamical systems in which time is *discrete* - $x_{n+1} = F(x_n;\mu)$. Such systems are called *maps*.

Why study maps?

- Maps can be used as tools for analyzing differential equations (e.g. Poincare maps for analysis of periodic solutions)
- Some phenomena are more naturally described by discrete time (e.g. some models in electronics, economy and finance, and even population dynamics)
- Maps are capable of more varied behavior than differential equations, an example of which in 1D we will see today, and hence may provide simple examples for complex mathematical behaviors. In particular, while in flows we have to go to higher dimensions to see chaos, some 1-dimensional maps exhibit chaos - more on this later in the course.

Additionally, maps are generally easy to simulate numerically due to their inherently discrete nature.

2 Analysis of fixed points

Reminder: in flows

A fixed point of the equation $\frac{dx}{dt} = \dot{x} = f(x)$ is a point x^* such that $f(x^*) = 0$.

For linear, one-dimensional flows, we saw:

$$\frac{dx}{dt} = \dot{x} = \lambda x \Rightarrow x(t) = x(0) \cdot e^{\lambda t} \tag{1}$$

So the stability of the fixed point $x^* = 0$ was determined by the value of λ - for $\lambda > 0$, for initial values near 0, we get exponential increase, and so the fixed point 0 is unstable, whereas for $\lambda < 0$ we get exponential decay, so the fixed point 0 is stable.

For nonlinear equations we used linearization near the fixed point: given x^* such that $f(x^*) = 0$, for values $x = x^* + \delta y$, we get

$$\dot{x} = \delta\dot{y} = f(x^* + \delta y) = f(x^*) + \delta y \cdot f'(x^*) + O(\delta^2) \tag{2}$$

So in the linear approximation,

$$\dot{y} = f'(x^*) \cdot y \tag{3}$$

Hence stability is determined by whether $f'(x^*) > 0$ (unstable fixed point), $f'(x^*) < 0$ (stable fixed point) or $f'(x^*) = 0$ (linear stability has failed - further analysis is required).

Fixed points in maps

Let $x_{n+1} = F(x_n)$. A *fixed point of the map* x^* is a point such that $F(x^*) = x^*$ (notice the difference from flows!). Indeed, if $x_n = x^*$ then $x_{n+1} = F(x_n) = F(x^*) = x^*$, so the orbit remains at x^* all future iterations.

To determine the stability of x^*, we consider a nearby orbit - $x_n = x^* + \delta_n$, and study its behavior - does the deviation δ_n grow or decay as n increases? Substituting into the map, we get:

$$x^* + \delta_{n+1} = x_{n+1} = F(x^* + \delta_n) = F(x^*) + F'(x^*)\delta_n + O(\delta_n^2) \tag{4}$$

This equation reduces to

$$\delta_{n+1} = F'(x^*)\delta_n + O(\delta_n^2) \tag{5}$$

Assume we can neglect the higher order terms. We remain with the linearized equation, $\delta_{n+1} = F'(x^*)\delta_n$, with the *eigenvalue* or ***multiplier*** $\lambda = F'(x^*)$. What are the dynamics of the linearized map?

$$\delta_1 = \lambda\delta_0, \delta_2 = \lambda\delta_1 = \lambda^2\delta_0, ..., \delta_n = \lambda^n\delta_0$$

We can therefore see that if $|\lambda| = |F'(x^*)| < 1$, then $\delta_n \to 0$ as $n \to \infty$ and the fixed point x^* is **linearly stable**. If $|F'(x^*)| > 1$ the fixed point is **unstable**. In the marginal case $|F'(x^*)| = 1$ linearization is not enough to determine stability, and we need to study the higher order terms.

A remark on oscillations

When studying flows in 1D, we had only monotone types of behavior - either converging to a fixed point or diverging from it. In maps, notice that when the multiplier is negative, we get oscillations, e.g. $x_{n+1} = -\frac{1}{2}x_n, x_0 = 1$ - we have $x_1 = -\frac{1}{2}, x_2 = \frac{1}{4}, x_3 = -\frac{1}{8}...$

Already we can see that 1D maps exhibit richer dynamics than 1D flows.

Example

Find the fixed points for the map $x_{n+1} = x_n^2$ and determine their stability.

$$x^* = (x^*)^2 \Rightarrow x^* = 0, 1$$

The multiplier is $\lambda = F'(x^*) = 2x^*$, therefore $x^* = 0$ is stable ($|\lambda| = 0 < 1$), and $x^* = 1$ is unstable ($|\lambda| = 2 > 1$).

3 Graphical analysis - cobweb diagrams

Similar to graphical analysis of fixed points of flows, we can also analyze the dynamics of a map graphically. We do this by plotting x_{n+1} vs. x_n, and considering the evolutions of trajectory by projecting to the diagonal. Such diagrams are called cobweb diagrams.

Construction: Given $X_{n+1} = F(x_n)$ and an initial condition X_0, draw a vertical line until it intersects the graph of F; that height is the output x_1. Now, trace a horizontal line till it intersects the diagonal line $x_{n+1} = x_n$ and then move vertically to the curve again. Repeat the process n times to generate the first n points in the orbit.

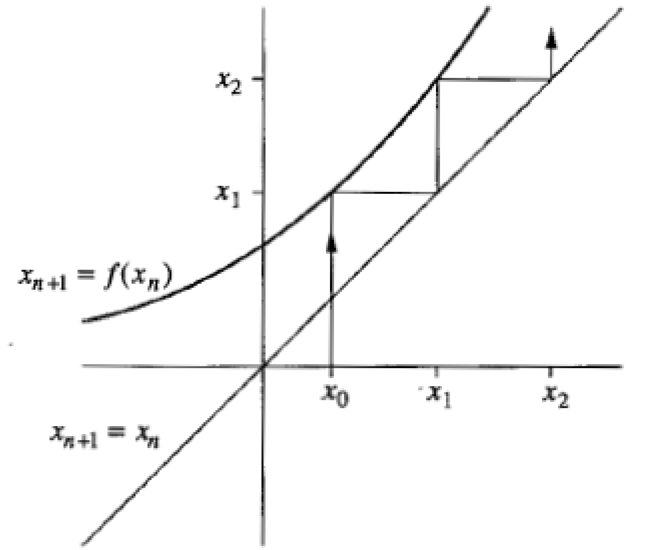
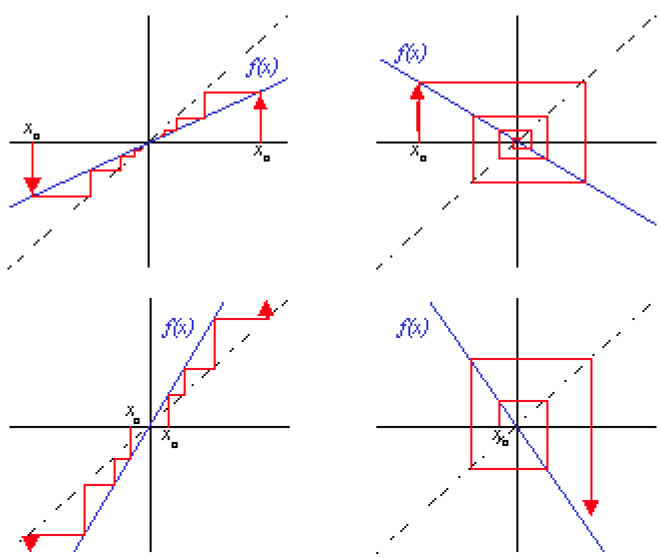


Figure 10.1.1

We consider the cobweb diagrams of linear maps. What happens when we are at the marginal case $x_{n+1} = -x_n$?



4 Bifurcations in 1D maps

Similar to what we have seen in class for flows, maps depending on parameters may also undergo bifurcations, i.e. a qualitative change in their behavior following a small parameter change. Recall, for flows we saw three types of co-dimension 1 bifurcations - the saddle-node bifurcation, the transcritical bifurcation, and the pitchfork bifurcation.

4.1 The Logistic Map

Consider the following map:

$$x_{n+1} = rx_n(1 - x_n) \tag{6}$$

This map is a discrete time analog of the logistic equation for population growth: x_n is a dimensionless measure of the population in the $n-$th generation (normalized by saturated population value) and $r \geq 0$ is the intrinsic growth rate. Note that our domain of allowed x values is $x \in \mathbb{R}^+$. $F(x)$ is a parabola with maximum value of $r/4$ at $x = \frac{1}{2}$. How do the dynamics change in dependence on the value of r? We restrict ourselves for now to the values $0 \leq r \leq 4$.

We study the fixed points:

$$x^* = F(x^*) = rx^*(1 - x^*) \Rightarrow x^* = 0 \text{ or } 1 = r(1 - x^*) \Rightarrow x^* = 1 - \frac{1}{r} \tag{7}$$

So the origin is a fixed point for all r, whereas $x^* = 1 - \frac{1}{r}$ is in our range of interest only if $r \geq 1$.

We analyze stability. The multiplier:

$$F'(x^*) = r - 2rx^*$$

For $x^* = 0$ we see that $F'(0) = r$, and hence the origin is stable for $r < 1$ and unstable for $r > 1$. At the other fixed point, $F'(x^*) = r - 2r(1 - \frac{1}{r}) = 2 - r$, so it is stable for $-1 < 2 - r < 1$, i.e. for $1 < r < 3$. It is unstable for $r > 3$.

One phenomenon we see here is at $r = 1$ when $x^* = 0$ loses stability. This is an example of a *transcritical bifurcation*.

However, our analysis also suggests that $x^* = 1 - \frac{1}{r}$ also loses its stability when $r = 3$, i.e. the system undergoes another bifurcation. This happens when the multiplier of x^* attains the critical value $F'(x^*) = -1$. The resulting bifurcation is called a *flip bifurcation*.

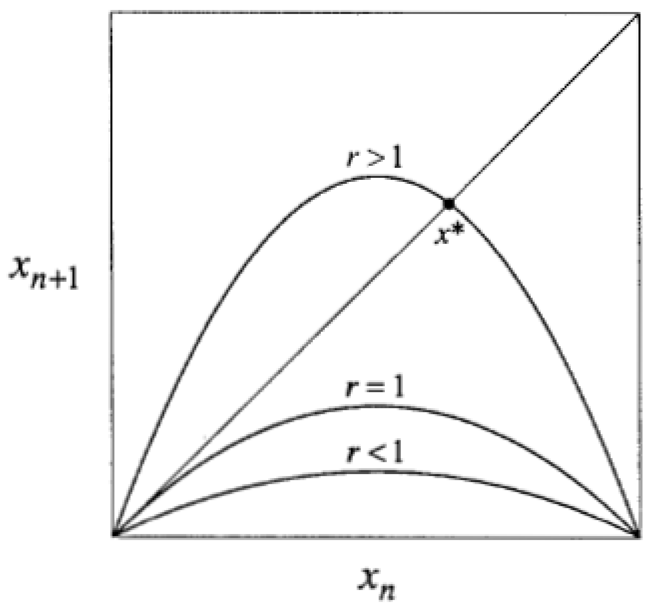


Figure 1: The logistic map for small r

Flip bifurcations are often associated with period-doubling, and here this is in fact the case. In order to prove that the logistic map has a 2-cycle for all $r > 3$, we use the following useful fact: Such a 2-cycle exists if and only if there exist p, q such that $F(q) = p$ and $F(p) = q$. Equivalently, p, q must satisfy that $F(F(p)) = p, F(F(q)) = q$. Hence p, q are *fixed points of the second-iterate map* $F^2(x) \equiv F(F(x))$. In the HW assignment you will find these roots algebraically - here we will show a graphic demonstration.

Bibliography

- Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering by Steven H. Strogatz
- https://www.mi.sanu.ac.rs/vismath/stewart/index.html

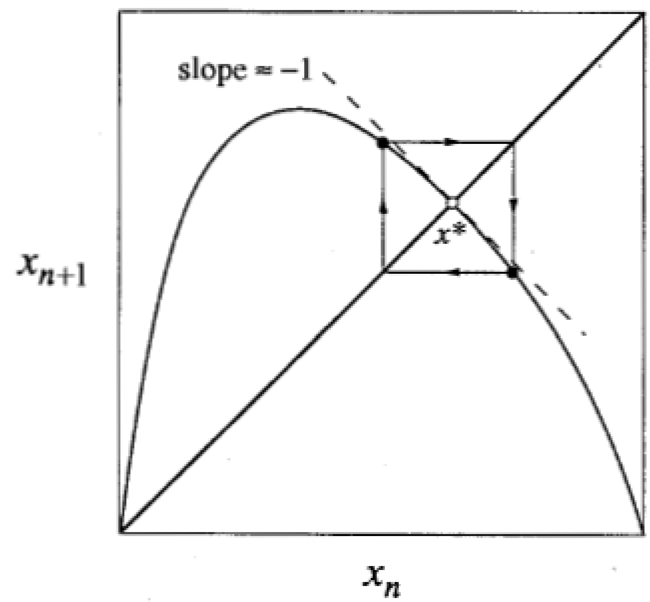


Figure 10.3.3

Figure 2: The logistic map - a periodic orbit is created

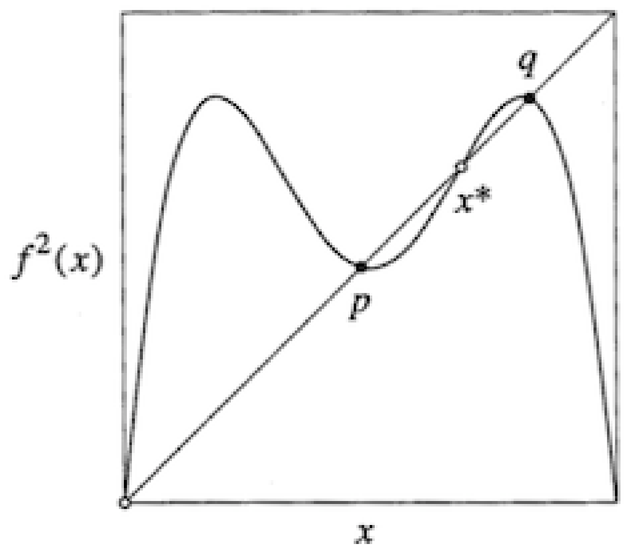


Figure 10.3.2

Figure 3: The logistic map - fixed points of F^2, $r > 3$